Propagation of light in single axis media

Let's first remember,

1) $\varepsilon_x = \varepsilon_y = \varepsilon_z$

Then the medium is isotropic and $\bar{E} \parallel \bar{D}$.
These are gases, liquids, cubic symmetry crystals, like NaCl, etc.

2) $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$ → single axis crystal

quartz, icelandic spar

3) $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$

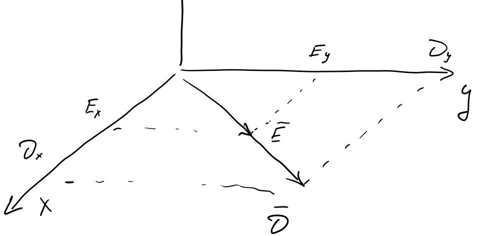
mica

Let's focus now on single axis

$$\begin{cases} \varepsilon_x = \varepsilon_y = \varepsilon_\perp \\ \varepsilon_z = \varepsilon_\parallel \end{cases}$$

① Let's direct light along z axis, i.e.
$\bar{s} \parallel z$

$$\boxed{\varepsilon_0 \left(\frac{c}{u}\right)^2 \bar{E} + \bar{s}(\bar{s}\,\bar{D}) - \bar{D} = 0} \quad *$$



Few preparations:

$$\bar{S} \perp \bar{E} \Rightarrow E_z = 0 \Rightarrow D_z = \varepsilon_0 \varepsilon_\parallel E_z = 0$$

$$\Downarrow$$

$$\bar{S} \perp \bar{D} \Rightarrow \bar{D}, \bar{E} \in xy$$

Since $\begin{cases} D_x = \varepsilon_0 \varepsilon_\perp E_x \\ D_y = \varepsilon_0 \varepsilon_\perp E_y \end{cases} \Rightarrow \bar{D} \parallel \bar{E}$

Equation $*$ in projection to x:

$$\varepsilon_0 \left(\frac{c}{u}\right)^2 E_x - \varepsilon_0 \varepsilon_\perp E_x = 0$$

in projection to y:

$$\varepsilon_0 \left(\frac{c}{u}\right)^2 E_y - \varepsilon_0 \varepsilon_\perp E_y = 0$$

It means, that

1) E_x and E_y are random
2) $u = \frac{c}{\sqrt{\varepsilon_\perp}}$

That means along z axis the wave of *any* polarization can propagate. This wave will propagate with speed

$$u = \frac{c}{\sqrt{\varepsilon_\perp}}$$

This direction is called crystal optical axis, the axis, along which we do not observe birefringence.

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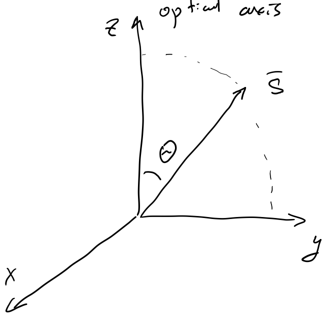
② Let's consider now the case of random direction of $\bar{s}$.



Plane of $\bar{s}$ and $\bar{z}$ is called main crystal cross-section.

If $y \in \bar{s}\bar{z}$, $x \perp y$

Again, let's remember

$$\varepsilon_0 \left(\frac{c}{u}\right)^2 \bar{E} + \bar{s}\left(\bar{s}\bar{D}\right) - \bar{D} = 0 \quad *$$

Let's make few preparations:

1) $\widehat{\bar{S}\, z} = \theta \Rightarrow S(0, \sin\theta, \cos\theta)$

2) $(\bar{S}\bar{D}) = \sin\theta\, D_y + \cos\theta\, D_z$

$$\begin{cases} D_x = \varepsilon_0 \varepsilon_\perp E_x \\ D_y = \varepsilon_0 \varepsilon_\perp E_y \\ D_z = \varepsilon_0 \varepsilon_\perp E_z \end{cases}$$

Hence, we have the following projections:

$$\begin{cases} \left[\left(\frac{c}{u}\right)^2 - \varepsilon_\perp\right] E_x = 0 \\ \left[\left(\frac{c}{u}\right)^2 - \varepsilon_\perp \cos^2\theta\right] E_y + \varepsilon_\parallel \cos\theta \sin\theta E_z = 0 \quad (\ast\ast) \\ \varepsilon_\perp \cos\theta \sin\theta E_y + \left[\left(\frac{c}{u}\right)^2 - \varepsilon_\parallel \sin^2\theta\right] E_z = 0 \end{cases}$$

This is system of homogeneous equations with argument E_x, E_y and E_z. In order to have solution to that equations:

$$\det = 0$$

det of such system will be:

$$\begin{vmatrix} \left(\frac{c}{u}\right)^2-\varepsilon_\perp & 0 & 0 \\ 0 & \left(\frac{c}{u}\right)^2-\varepsilon_\perp\cos^2\theta & \varepsilon_\parallel\cos\theta\sin\theta \\ 0 & \varepsilon_\perp\cos\theta\sin\theta & \left(\frac{c}{u}\right)^2-\varepsilon_\parallel\sin^2\theta \end{vmatrix}=0$$

Let's find its roots.

Root 1: From first row

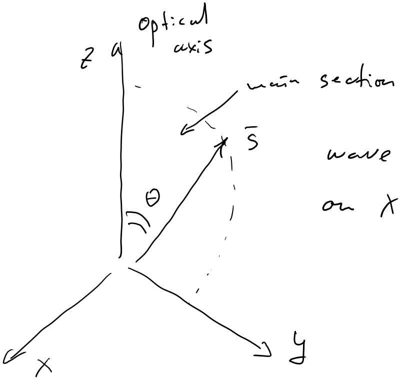
$$\left(\frac{c}{u}\right)^2-\varepsilon_\perp=0 \qquad u=\frac{c}{\sqrt{\varepsilon_\perp}}=\frac{c}{n_o}$$

n_o is refractive index of ordinary wave

Now, what will be the polarization of this wave?

If we place it in equation 1 of (**), then

$$\left[\left(\frac{c}{u}\right)^2-\varepsilon_\perp\right]E_x=0 \Rightarrow E_x \text{ is random}$$



That means along $\bar{S}$ we can have wave with any projections on X.

What will be with E_z. Let's place root 1 into 2 and 3 equations of $(**)$:

$$\begin{cases} \left[\varepsilon_\perp - \varepsilon_\perp \cos^2\theta\right] E_y + \varepsilon_\parallel \cos\theta \sin\theta \, E_z = 0 \\ \varepsilon_\parallel \cos\theta \sin\theta \, E_y + \left[\varepsilon_\perp - \varepsilon_\parallel \sin^2\theta\right] E_z = 0 \end{cases}$$

Again, this is system of homogeneous equations and non-trivial solution will be only, when $\det = 0 \Rightarrow \varepsilon_\parallel = \varepsilon_\perp$

However, this is not possible, otherwise it would be isotropic medium.

Pleas, solve it

Thus, the only solution would be, when:

$$E_y = E_z = 0$$

What does it mean?

Along random direction $\bar{s}$ we have propagating wave, that is polarized perpendicular to main section and propagates with speed independent of direction.

We only looked at root 1. But we also have root 2 and 3.

Root 2:

$$\begin{vmatrix} \left(\frac{c}{u}\right)^2 - \varepsilon_\perp \cos^2\theta & \varepsilon_\parallel \cos\theta \sin\theta \\ \varepsilon_\perp \cos\theta \sin\theta & \left(\frac{c}{u}\right)^2 - \varepsilon_\parallel \sin^2\theta \end{vmatrix} = 0$$

We will get the following equation:

$$\left(\frac{c}{u}\right)^4 - \left(\frac{c}{u}\right)^2 \left(\varepsilon_\parallel \sin^2\theta + \varepsilon_\perp \cos^2\theta\right) - \varepsilon_\parallel \varepsilon_\perp \sin^2\theta \cos^2\theta +$$

$$+ \varepsilon_\parallel \varepsilon_\perp \sin^2\theta \cos^2\theta = 0$$

Hence,

$$u = \frac{c}{}$$

$$\sqrt{\varepsilon_{\parallel} \sin^2\theta + \varepsilon_{\perp} \cos^2\theta}$$

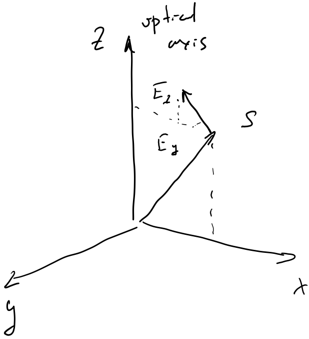
Very interesting, ray velocity depends on θ.

Again, let's put this roots 2 and 3 into equation 1 of (**).

$$(\varepsilon_{\parallel} \sin^2\theta + \varepsilon_{\perp} \cos^2\theta - \varepsilon_{\perp}) E_x = 0$$

$$(\varepsilon_{\parallel} - \varepsilon_{\perp}) E_x = 0 \Rightarrow E_x = 0$$

If we put roots 2 and 3 into equations 2 and 3 of (**), we will find E_z and E_y components.



$$\sin\theta\, E_y + \cos\theta\, E_z = 0$$

$$\text{tg}\,\theta = -\frac{E_z}{E_y}$$

What does it mean?

Along direction $\bar{s}$ we have second wave, that have

$$\begin{cases} E_x = 0 \\ tg\,\theta = -\dfrac{E_z}{E_y} \end{cases}$$

It will be polarized in the plane of main section and its speed will depend on the direction Θ.

That means we will have two waves along $\bar{s}$.