Propagation of light in single axis media

Let's first remember,

1) $\vec{B}_0 = \vec{S}_0 \times \vec{E}_0$

Since the medium is isotropic and $\vec{E} \parallel \vec{D}$ There are gases, liquids, cubic symmetry crystals, like NaCl, etc.

2) $\vec{B}_0 \times \vec{E}_0 \times \vec{E}_0 \Rightarrow$ triple cross original quantity, isotropic space

3) $\vec{B}_0 \neq \vec{E}_0 \neq \vec{E}_0$

in-case

Let's focus more on single axis

$$\begin{cases} \vec{B}_0 = \vec{E}_0 = \vec{E}_1 \\ \vec{E}_0 = \vec{E}_0 \end{cases}$$

① Let's direct light along z axis, i.e. $\vec{S} \parallel \vec{z}$

$$\boxed{\vec{E}_0 \left(\frac{c}{n_0}\right)^2 \vec{E} - \vec{S}(\vec{S}\cdot\vec{D}) - \vec{D} = 0} \quad *$$

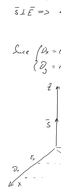
For perpendicular

$\vec{S} \perp \vec{E} \Rightarrow \vec{E}_0 = 0 \Rightarrow \vec{D}_0 = \vec{E}_0 \vec{E}_0 = 0$

$\vec{S} \perp \vec{D} \Rightarrow \vec{D} \perp \vec{E} \Rightarrow \vec{D}$

Since $\vec{D}_0 = \vec{E}_0 \vec{E}_0 \Rightarrow \vec{D} \parallel \vec{E}$

$\vec{D}_0 = \vec{E}_0 \vec{E}_0$



Equation $*$ is projection to x:

$$\vec{E}_0 \left(\frac{c}{n_0}\right)^2 \vec{E}_0 - \vec{E}_0 \vec{E}_0 \vec{E}_0 = 0$$

is projection to y:

$$\vec{E}_0 \left(\frac{c}{n_0}\right)^2 \vec{E}_0 - \vec{E}_0 \vec{E}_0 \vec{E}_0 = 0$$

It means, that

1) $\vec{E}_0$ and $\vec{E}_0$ are random

2) $u = \frac{c}{\sqrt{\vec{E}_0}}$

That means along z axis the wave of any polarization can propagate. This wave will propagate with speed

$$u = \frac{c}{\sqrt{\vec{E}_0}}$$

The direction is called crystal optical axis, the axis, along which we do not observe birefringence.

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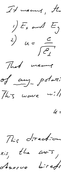
② Let's consider now the case of random direction of $\vec{S}$.



Plane of $\vec{S}$ and $\vec{z}$ called main crystal cross-section.

If $\vec{y} \in \vec{S}\vec{z}$, $x \parallel y$

Again, let's remember

$$\vec{E}_0 \left(\frac{c}{n_0}\right)^2 \vec{E} + \vec{S}(\vec{S}\cdot\vec{D}) - \vec{D} = 0 \quad *$$

Let's make few projections:

1) $\vec{S}\vec{E} = 0 \Rightarrow S(0, S_y = 0, \cos\theta)$

2) $(\vec{S}\vec{D}) = S_x D_x + S_y D_y = S_z D_z$

$$\begin{cases} D_x = \vec{E}_0 \vec{E}_x \\ D_y = \vec{E}_0 \vec{E}_y \\ D_z = \vec{E}_0 \vec{E}_z \end{cases}$$

Hence, we have the following projections:

$$\begin{cases} \left[\left(\frac{c}{n_0}\right)^2 - \vec{E}_0\right] E_x = 0 \\ \left[\left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \cos^2\theta\right] E_y + \vec{E}_0 \cos\theta \sin\theta E_z = 0 \quad (\text{for } \vec{S}\vec{E}) \\ \vec{E}_0 \cos\theta \sin\theta E_y + \left[\left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \sin^2\theta\right] E_z = 0 \end{cases}$$

This is system of homogeneous equations with unknowns E_x, E_y and E_z. In order to have solution to this equations:

$$\det \neq 0$$

det of such system will be:

$$\begin{vmatrix} \left(\frac{c}{n_0}\right)^2 - \vec{E}_0 & 0 & 0 \\ 0 & \left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \cos^2\theta & \vec{E}_0 \cos\theta \sin\theta \\ 0 & \vec{E}_0 \cos\theta \sin\theta & \left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \sin^2\theta \end{vmatrix} = 0$$

Let's find its roots.

Root 1: from first row

$$\left(\frac{c}{n_0}\right)^2 - \vec{E}_0 = 0 \qquad u = \frac{c}{\sqrt{\vec{E}_0}} = \frac{c}{n_0}$$

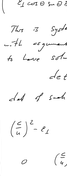
n_0 is refractive index of ordinary wave

Now, what will be the polarization of this wave?

If we place it in equation 1 of $(*)$, then

$$\left[\left(\frac{c}{n_0}\right)^2 - \vec{E}_0\right] E_x = 0 \Rightarrow E_x \text{ is random}$$

That means along



main section $\vec{S}$ we can have waves with any polarization on x.

What will be with E_y. Let's place root 1 into 2 and 3 equations of $(*)$.

$$\begin{cases} \left[\vec{E}_0 - \vec{E}_0 \cos^2\theta\right] E_y + \vec{E}_0 \cos\theta \sin\theta E_z = 0 \\ \vec{E}_0 \cos\theta \sin\theta E_y + \left[\vec{E}_0 - \vec{E}_0 \sin^2\theta\right] E_z = 0 \end{cases}$$

Again, this is system of homogeneous equations and non-trivial solutions will be only, when $\det = 0 \Rightarrow \vec{E}_y = \vec{E}_z$

However, this is not possible (Plane sides of) otherwise, it would be isotropic medium

Thus, the only solution would be, when:

$$\vec{E}_y = \vec{E}_0 = 0$$

What does it mean?

Along random direction $\vec{S}$ we have propagating wave, that is polarized perpendicular to main section and propagates with speed independent of direction.

We only looked at root 1. But we also have root 2 and 3

Root 2:

$$\begin{vmatrix} \left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \cos^2\theta & \vec{E}_0 \cos\theta \sin\theta \\ \vec{E}_0 \cos\theta \sin\theta & \left(\frac{c}{n_0}\right)^2 - \vec{E}_0 \sin^2\theta \end{vmatrix} = 0$$

We will get the following equation:

$$\left(\frac{c}{n_0}\right)^2 \left(\frac{c}{n_0}\right)^2 \vec{E}_0 \sin^2\theta + \vec{E}_0 \cos^2\theta \cdot \vec{E}_0 \sin^2\theta - \left(\frac{c}{n_0}\right)^2 \vec{E}_0 \cos^2\theta - \vec{E}_0 \vec{E}_0 \sin^2\theta \cdot \vec{E}_0 \cos^2\theta = 0$$

Hence,

$$u = \frac{c}{\sqrt{\vec{E}_0 \sin^2\theta + \vec{E}_0 \cos^2\theta}}$$

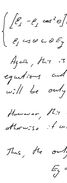
Very interesting, ray velocity depends on θ

Again, let's put this root into equation 1 of $(*)$.

$$\left(\vec{E}_0 \sin^2\theta + \vec{E}_0 \cos^2\theta - \vec{E}_0\right) E_x = 0$$

$$\left(\vec{E}_0 - \vec{E}_0\right) E_x = 0 \Rightarrow E_x = 0$$

If we put root into equations 2 and 3 of $(*)$, we will find E_y and E_z components.



$$\sin\theta E_y + \cos\theta E_z = 0$$

$$\operatorname{tg}\theta = -\frac{E_z}{E_y}$$

What does it mean?

Along direction $\vec{S}$ we have second wave, that have

$$\begin{cases} E_x = 0 \\ \operatorname{tg}\theta = -\frac{E_z}{E_y} \end{cases}$$

It will be polarized in the plane of main section and its speed will depend on the direction θ

That means we will have two waves along $\vec{S}$.